

DEFORMATIONS AND NONLINEAR SYSTEMS

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Abstract

The q-deformation of harmonic oscillators is shown to lead to q-nonlinear vibrations. The examples of q-nonlinearized wave equation and Schrödinger equation are considered. The procedure is generalized to broader class of nonlinearities related to other types of deformations. The nonlinear noncanonical transforms used in the deformation procedure are shown to preserve in some cases the linear dynamical equations, for instance, for the harmonic oscillators. The nonlinear coherent states and some physical aspects of the deformations are reviewed.

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1 Introduction

We will review an approach to some nonlinear dynamical systems related to deformations of linear classical and quantum systems [1, 2]. The idea of the approach is to replace parameters of a linear system with constants of the motion of a nonlinear system. This procedure produces from a linear system a nonlinear one and the q-oscillator of [3, 4] may be considered as physical system with this specific nonlinearity. For q-oscillator, the constant parameter which was replaced by the constant of the motion depending on the amplitude of the vibration was the frequency.

The approach may be used for many dynamical systems. So, the constant masses in Klein-Gordon and Dirac equations or the constant signal velocity in the wave equation may be replaced by dynamical variables but these variables are chosen to be constants of the motion for the dynamical system under consideration. In fact, we address the problem whether physical constant parameters like light velocity, Planck constant, or gravitational constant are constant parameters in reality or they may depend, for example, on the initial condition in the process of evolution? The same question may be put about such characteristics of elementary particle as electric charge or mass, say, of electron or fine structure constant which play the role of constant parameters in the dynamical equations of the theory. The corresponding nonlinear system of equations obtained in this approach may be simply a “reparametrized” initial linear system of equations in which constant parameters are replaced by constants of the motion of the nonlinear system.

One could deform also a simple initial nonlinear system containing some constant parameters and to transform it into another nonlinear system by replacing the constants with the integrals of the motion of the nonlinear system. So, it is possible to extend the class of integrable nonlinear systems starting from simple integrable nonlinear systems and reparametrizing them, i.e., replacing

the constant parameters by the constants of the motion.

The aim of our work is to give a review of the approach to q -deformation based on the nonlinearization procedure and to present a general scheme for the described procedure of deforming the linear systems. We will consider examples of q -deformed harmonic oscillator, q -deformed wave equation, and q -deformed Schrödinger equation.

The ansatz of deformation used in our approach is reduced to applying a nonlinear noncanonical transformation to oscillator complex amplitude. This transformation adds to the complex amplitude a new factor which is integral of the motion for the oscillator motion both in linear and nonlinear regimes of vibrations. Having this observation will discuss, about compatibility of system dynamics with different commutation relations for the observables.

We will review the results of [6], in which it was shown that one and the same harmonic oscillator dynamics is compatible with different commutation relations of the oscillator quadratures. The nonlinear transformation of the oscillator amplitude by adding to it the factor which is constant of the motion leads naturally to the notion of nonlinear coherent states [7] of f -coherent states [8]. Thus, we will review the properties of these states for which q -coherent states [3] are the partial case.

The physical consequences of the q -nonlinearity as blue shift effect [9], deformation of Planck distribution formula [9, 10], and change of a charge form-factor [11] will be discussed, as well as constructions of nonlinear coherent states [7] (f -coherent states [8]).

2 Quantum q -Oscillator

The usual creation and annihilation oscillator operators a and a^\dagger obeying bosonic commutation

relations

$$[a, a^\dagger] = 1 \quad (1)$$

have in the Fock basis the known expressions

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ \dots & \dots & \dots & \dots \end{pmatrix}; \quad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (2)$$

The q -oscillators may be introduced by generalizing the matrices (2) with the help of the q -integer numbers n_q ,

$$n_q = \frac{\sinh n\lambda}{\sinh \lambda}; \quad q = e^\lambda. \quad (3)$$

Here λ and q are dimensionless c -numbers. At $\lambda = 0$, $q = 1$ and the q -integer n_q coincides with n . Replacing the integers in (2) by q -integers we obtain matrices which define the annihilation and creation operators of the quantum q -oscillator,

$$a_q = \begin{pmatrix} 0 & \sqrt{1_q} & 0 & \dots \\ 0 & 0 & \sqrt{2_q} & \dots \\ 0 & 0 & 0 & \sqrt{3_q} \\ \dots & \dots & \dots & \dots \end{pmatrix}; \quad a_q^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1_q} & 0 & 0 & \dots \\ 0 & \sqrt{2_q} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (4)$$

The above matrices obey the commutation relation

$$[a_q, a_q^\dagger] = F(\hat{n}); \quad \hat{n} = a_q^\dagger a_q, \quad (5)$$

where the function $F(\hat{n})$ has the form

$$F(\hat{n}) = \frac{\sinh \lambda(\hat{n} + 1) - \sinh \lambda \hat{n}}{\sinh \lambda}. \quad (6)$$

In addition, there exists the reordering relation

$$a_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} - a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} = \mathbf{q}^{-\hat{n}}. \quad (7)$$

The operators $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^{\dagger}$ can be expressed in terms of the operators a and a^{\dagger}

$$a_{\mathbf{q}} = a f(\hat{n}); \quad a_{\mathbf{q}}^{\dagger} = f(\hat{n}) a^{\dagger}, \quad (8)$$

where

$$f(\hat{n}) = \sqrt{\frac{a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}}{a^{\dagger} a}}. \quad (9)$$

The classical harmonic oscillator vibrating with unit frequency may be described in terms of complex variables

$$\alpha = \frac{q + ip}{\sqrt{2}}; \quad \alpha^* = \frac{q - ip}{\sqrt{2}}, \quad (10)$$

with nonzero Poisson bracket

$$\{\alpha, \alpha^*\} = -i. \quad (11)$$

Defining the classical \mathbf{q} -oscillator in terms of new variables

$$\alpha_{\mathbf{q}} = \sqrt{\frac{\sinh \lambda \alpha \alpha^*}{\alpha \alpha^* \sinh \lambda}} \alpha; \quad \alpha_{\mathbf{q}}^* = \sqrt{\frac{\sinh \lambda \alpha \alpha^*}{\alpha \alpha^* \sinh \lambda}} \alpha^*, \quad (12)$$

we get the Poisson brackets

$$\{\alpha_{\mathbf{q}}, \alpha_{\mathbf{q}}^*\} = -i \frac{\lambda}{\sinh \lambda} \sqrt{1 + |\alpha_{\mathbf{q}}|^4 (\sinh \lambda)^2}. \quad (13)$$

We will consider a new system described by such \mathbf{q} -variables with the Hamiltonian function

$$H(\alpha_{\mathbf{q}}, \alpha_{\mathbf{q}}^*) = \alpha_{\mathbf{q}} \alpha_{\mathbf{q}}^*. \quad (14)$$

Then the equations of the motion are

$$\dot{\alpha}_{\mathbf{q}} = -i \frac{\lambda}{\sinh \lambda} \sqrt{1 + |\alpha_{\mathbf{q}}|^4 (\sinh \lambda)^2} \alpha_{\mathbf{q}} \quad (15)$$

with solutions

$$\alpha_q(t) = \alpha_q(0) \exp \left[\frac{-it\lambda}{\sinh \lambda} \sqrt{1 + |\alpha_q(0)|^4 (\sinh \lambda)^2} \right]. \quad (16)$$

We have performed a noncanonical transformation, i.e., deformed the Poisson bracket, while preserving the form of the Hamiltonian. Such a system can be rewritten in terms of (α, α^*) variables: in these coordinates, the original Poisson bracket is unchanged while the Hamiltonian is

$$H_q(\alpha, \alpha^*) = \frac{\sinh \lambda \alpha \alpha^*}{\sinh \lambda}. \quad (17)$$

This dynamical system has a phase portrait which is the same as the usual linear harmonic oscillator. The equations of the motion for the system with the Hamiltonian H_q are

$$\dot{\alpha} = -i\omega_q \alpha; \quad \dot{\alpha}^* = i\omega_q \alpha^* \quad (18)$$

with

$$\omega_q \equiv \omega_q(\alpha \alpha^*) = \frac{\lambda}{\sinh \lambda} \cosh \lambda \alpha \alpha^*. \quad (19)$$

We notice that $\alpha \alpha^*$ is a constant of the motion for the system, the frequency of which depends on the orbit.

This leads to interpreting q-oscillators as systems carrying a particular nonlinearity. What in the harmonic oscillator is a constant frequency characterizing the evolution along any orbit, becomes a function constant on each orbit, separately. For the q-oscillator, the frequency depends on amplitude of vibrations.

The nonlinear second order equation for the coordinate of the oscillator q is

$$\ddot{q} + \omega_q^2 q = 0. \quad (20)$$

It means that the dynamics of the coordinate q and momentum p of the nonlinear q -oscillator is described by the system of equations

$$\dot{q} = \omega_q(\alpha\alpha^*)p; \quad \dot{p} = -\omega_q(\alpha\alpha^*)q, \quad (21)$$

where

$$\alpha\alpha^* = \frac{1}{2}(q^2 + p^2). \quad (22)$$

For the deformed equations of the motion of the q -oscillator, in view of (21), the momentum is the function of the velocity and position. This function may be obtained as the solution to the functional equation

$$p(q, \dot{q}) = \frac{\sinh \lambda}{\lambda} \frac{\dot{q}}{\cosh \{(\lambda/2)[q^2 + p^2(q, \dot{q})]\}}, \quad (23)$$

considered as the implicit formula for the giving momentum as the function of the position q and velocity \dot{q} . The solution to the q -oscillator equation of the motion

$$\dot{\alpha} = -i\alpha \frac{\lambda}{\sinh \lambda} \cosh \lambda\alpha\alpha^* \quad (24)$$

is

$$\alpha(t) = \alpha_0 \exp \left[-it \frac{\lambda}{\sinh \lambda} \cosh \lambda\alpha_0\alpha_0^* \right], \quad (25)$$

where

$$\alpha_0 = \alpha(t=0)$$

is the initial complex amplitude of the nonlinear q -oscillator. The solution to the equation of the motion for the coordinate q of the nonlinear q -oscillator

$$\ddot{q} + \frac{\lambda^2}{\sinh^2 \lambda} \cosh^2 \left\{ \frac{\lambda}{2} [q^2 + p^2(q, \dot{q})] \right\} = 0, \quad (26)$$

where the function $p(q, \dot{q})$ is given implicitly by relation (23), may be written in the form

$$\begin{aligned}
q(t) = & \frac{q_0}{2} \left\{ \exp \left[\frac{i\lambda t}{\sinh \lambda} \cosh \left\{ \frac{\lambda}{2} [q_0^2 + p^2(q_0, \dot{q}_0)] \right\} \right] \right. \\
& + \exp \left[-\frac{i\lambda t}{\sinh \lambda} \cosh \left\{ \frac{\lambda}{2} [q_0^2 + p^2(q_0, \dot{q}_0)] \right\} \right] \left. \right\} \\
& + \frac{\dot{q}_0 \sinh \lambda}{2i\lambda} \cosh^{-1} \left\{ \frac{\lambda}{2} [q_0^2 + p^2(q_0, \dot{q}_0)] \right\} \\
& \times \left\{ \exp \left[\frac{i\lambda t}{\sinh \lambda} \cosh \left\{ \frac{\lambda}{2} [q_0^2 + p^2(q_0, \dot{q}_0)] \right\} \right] \right. \\
& - \exp \left[-\frac{i\lambda t}{\sinh \lambda} \cosh \left\{ \frac{\lambda}{2} [q_0^2 + p^2(q_0, \dot{q}_0)] \right\} \right] \left. \right\}. \tag{27}
\end{aligned}$$

Here $q_0 = q(t=0)$ and $\dot{q}_0 = \dot{q}(t=0)$ are the initial position and velocity of the nonlinear q -oscillator. In the limit $\lambda \rightarrow 0$, we have the standard solution for the linear harmonic oscillator.

One can find for small nonlinearity $\lambda \ll 1$ the approximate expression for the momentum solving Eq. (23) by iteration method. We have

$$p = \dot{q} \left[1 + \frac{\lambda^2}{6} - \frac{\lambda^2}{8} (q^2 + \dot{q}^2) \right]. \tag{28}$$

Formula (28) may be interpreted as the negative shift of the oscillator mass by the factor depending quadratically on the energy of the oscillations.

3 Deformed Wave Equations

Following [2] we start from the wave equation of the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \varphi(x, t) = 0 \tag{29}$$

and represent this equation as a system of equations for decoupled oscillators. To do this, we rewrite Eq. (29) in momentum representation

$$\ddot{\varphi}(k, t) + k^2 \varphi(k, t) = 0, \tag{30}$$

where the complex Fourier amplitude

$$\varphi(k, t) = \frac{1}{2\pi} \int \varphi(x, t) \exp(-ikx) dx \quad (31)$$

plays the role of new coordinate. Since $\varphi(x, t) = \varphi^*(x, t)$, we have $\varphi(k, t) = \varphi^*(-k, t)$. Equation (30) describes a two-dimensional oscillator with equal frequencies for both modes labeled by k and $-k$. Writing Eq. (30) in the form

$$\dot{\varphi}(k, t) = \pi(k, t); \quad \dot{\pi}(k, t) = -k^2 \varphi(k, t), \quad (32)$$

we have the equations in phase spaces of the oscillators with nonunit frequency $\omega^2 = k^2$, $p \rightarrow \pi(k, t)$, and $q \rightarrow \varphi(k, t)$. Due to this, we can deform this linear system taking the integral of the motion

$$\mu = \int dk \left\{ \frac{1}{2|k|} [k^2 |\varphi|^2(k, t) + |F_k|^2] \right\}, \quad (33)$$

in which the function F_k playing the role of complex momentum of k -field mode is a solution to the infinite system of equations

$$\dot{\varphi}(k, t) = F_k f_Q \left\{ \int dk' \frac{1}{2|k'|} [k'^2 |\varphi|^2(k', t) + |F_{k'}|^2] \right\}. \quad (34)$$

The function f_Q is

$$f_Q(z) = \frac{\lambda}{\sinh \lambda} \cosh(\lambda z). \quad (35)$$

The parameter μ plays the role of the initial number of vibrations corresponding to given Cauchy initial conditions. Then the deformed wave equation may be written in the form

$$\ddot{\varphi}(x, t) = f_Q^2(\mu) \frac{\partial^2}{\partial x^2} \varphi(x, t). \quad (36)$$

We have the differential-functional equation which looks like standard wave equation with the wave velocity $f_Q(\mu)$ being a constant of the motion. Thus, the procedure of deformation yields us the

nonlinear equation for which the velocity of wave propagation depends on the initial configuration of the field and its time derivative.

The q -deformed wave equation (36) has the soliton-like solutions

$$\varphi_{\pm}(x, t) = \Phi\left(x \pm f_q(\mu) t\right), \quad (37)$$

where Φ is an arbitrary function. In fact, discussed q -deformation implies the existence of nonlinear interaction among the modes. The q -deformed Klein–Gordon equation is considered by this method in [12]. Some deformed relativistic equations are discussed in [13, 14] from different viewpoints.

4 Nonlinear Quantum Equation for One-Level System

Following [6] we start with a one-level quantum system to show that the Schrödinger equation for it gives rise to a Hamiltonian dynamics for a one-dimensional harmonic oscillator. In fact, the Schrödinger equation

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H \Psi(t), \quad (38)$$

where $\Psi(t)$ is the wave function of the one-level system, is described by the Hamiltonian H . This Hamiltonian is the Hermitian 1×1 -matrix and it means that the Hamiltonian is simply a c -number which is real.

If one introduces the two real variables $q(t)$ and $p(t)$ as real and imaginary parts of the wave function

$$\Psi(t) = \frac{1}{\sqrt{2}} \left[\frac{H}{\hbar} q(t) + i p(t) \right], \quad (39)$$

the Schrödinger equation acquires the form

$$\dot{q} = p; \quad \dot{p} = -\frac{H^2}{\hbar^2} q. \quad (40)$$

Then introducing the frequency $\omega = H/\hbar$, one can rewrite Eqs. (40) as

$$\ddot{q} + \omega^2 q = 0, \quad (41)$$

which represents the equation of the motion for the one-mode harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}. \quad (42)$$

We may call this system a classical-like system (the observation that quantum wave equation may be rewritten as classical-like equation was done in [15]).

The procedure used in Section 2 gives possibility to nonlinearize the linear Schrödinger equation. For one-level system, time units can be chosen in such a way that the frequency ω is equal to unity. Then, for example, the method of q-nonlinearization of Section 2 gives

$$i\dot{\Psi} = \left[\frac{\lambda}{\sinh \lambda} \cosh(\lambda|\Psi|^2) \right] \Psi. \quad (43)$$

Equation (43) is a nonlinear Schrödinger equation obtained in the frame of using q-nonlinearity of vibration of the complex probability amplitude. The procedure may be easily extended for multilevel quantum system. We simply apply the same procedure of the q-nonlinearization to the stationary state wave functions belonging to chosen energy levels, because real and imaginary parts of the stationary wave functions are just coordinates and momenta of the one-dimensional oscillators.

5 Quantum Oscillations and Commutation Relations

In [5], Wigner studied the following problem: To what extent do the equations of the motion determine the quantum mechanical commutation relations? He found that the commutation relations are not uniquely determined by the equations of the motion even if the form of the Hamiltonian is fixed. The problem was discussed in detail in [6] and below we review the results of this work.

The equation of the motion for the linear quantum oscillator, with the frequency and mass such that $\omega = m = 1$ and Planck constant is equal to unity, considered in [5] may be written in terms of complex amplitude operators a (annihilation operator) and a^\dagger (creation operator) as

$$\dot{a} + ia = 0; \quad \dot{a}^\dagger - ia^\dagger = 0. \quad (44)$$

The commutation relation (1) may be chosen for these two operators. The Hamiltonian describing the linear equation (44) may be taken in standard form

$$H = a^\dagger a + \frac{1}{2}. \quad (45)$$

Using Eq. (45) one can check that the Heisenberg equation of the motion for the complex amplitude operator a

$$i\dot{a} = [a, H] \quad (46)$$

yields Eq. (44).

Now we will show that there exists another alternative description pair, Hamiltonian–commutation relation, which produces the same dynamics. To do this, we first multiply Eq. (44) from the right-hand side by the operator $f(\hat{n})$, where $\hat{n} = a^\dagger a$ and the real function $f(x)$ has no zeros at nonnegative integers. Since the operator \hat{n} is the integral of the motion for the dynamics given

by Eq. (44), a function of the integral of the motion is also constant of the motion. Then we have for the operator

$$A = af(\hat{n}) \quad (47)$$

the evolution equation

$$\dot{A} + iA = 0. \quad (48)$$

The nonlinear transformation (47) preserves the linearity of the particular equation (44). Such transformations may be called *linearoid* in analogy with *canonoid* in classical mechanics, which are noncanonical transformations in general but, for particular case, preserve the symplectic structure (Poisson brackets). The linearoid (47) has inverse

$$a = A \frac{1}{f(\hat{n})}, \quad (49)$$

in which we have to use

$$A^\dagger = f(\hat{n}) a^\dagger \quad (50)$$

and

$$\hat{N} = A^\dagger A = f^2(\hat{n}) \hat{n} = F(\hat{n}) \quad (51)$$

to express the operator \hat{n} as the function of the operator N

$$\hat{n} = F^{-1}(\hat{N}). \quad (52)$$

The function $y = F^{-1}(x)$ is the inverse function of the relation $x = F(y)$. Relation (49) may be rewritten as

$$a = A \frac{1}{f[F^{-1}(\hat{N})]} \quad (53)$$

and also

$$a^\dagger = \frac{1}{f[F^{-1}(\hat{N})]} A^\dagger. \quad (54)$$

The operators A and A^\dagger satisfy the commutation relations

$$AA^\dagger - A^\dagger A = \varphi[F^{-1}(\hat{N})], \quad (55)$$

where the function $\varphi(z)$ is related to the function $f(z)$ from Eq. (47) by

$$\varphi(z) = (z+1)f^2(z+1) - zf^2(z). \quad (56)$$

It is obvious, that since we performed only a change of variables using the invertible nonlinear noncanonical transformation (47), the Hamiltonian for the variable A obeying the commutation relations (55) and the linear equation of the motion (48) is the same Hamiltonian (45) but expressed in terms of the variables A and A^\dagger , i.e.,

$$H = \hat{F}^{-1}(N) + \frac{1}{2}. \quad (57)$$

On the other hand, we know that we could define (with another Hilbert space structure) the operators B and B^\dagger with commutation relations

$$BB^\dagger - B^\dagger B = 1 \quad (58)$$

and with the Hamiltonian

$$H' = B^\dagger B + \frac{1}{2}, \quad (59)$$

obeying to the Heisenberg equations of the motion

$$\dot{B} + iB = 0 \quad \dot{B}^\dagger - iB^\dagger = 0. \quad (60)$$

Thus, we conclude that for one and the same dynamics, i.e., one and the same linear equations of the motion (48) and (60), there exists an infinite number of Hamiltonians and commutation relations giving the same dynamics. The ambiguity in choosing alternative Hamiltonians is labeled in the demonstrated procedure by a function f of the oscillator energy.

It is worthy to note that the commutation relation (55) implies that the dispersions of quadratures

$$P = \frac{1}{i\sqrt{2}}(A - A^\dagger) \quad \text{and} \quad Q = \frac{1}{\sqrt{2}}(A + A^\dagger)$$

do not satisfy the Heisenberg uncertainty relation. But according to the dynamical equation (48) these quadratures evolve performing usual harmonic oscillations.

It is known [16, 17], that there exists an infinite number of variational formulations for one-dimensional classical systems. It is possible to use this alternative Hamiltonian descriptions to construct alternative quantum descriptions for the oscillator [6].

6 One-Mode f-Coherent States

Coherent states were originally introduced as eigenstates of the annihilation operator for the harmonic oscillator and then widely used in physics, particularly in quantum optics. This is therefore a concept of algebraic origin and having now constructed a similar annihilation operator it is natural, following the same procedure, to construct a new class of f-coherent states (nonlinear coherent states) in the Fock space. Further the f-coherent states may not be preserved under time evolution. Nevertheless, we are willing to call them “f-coherent states” for an easy identification, of the kind already proposed for the eigenstates of the q-annihilation operator which were named “q-coherent states” [3].

Let us take for the one-mode case the operator (47). Then one can consider the eigenfunctions of A , $|\alpha, f\rangle$ in a Hilbert space. They therefore satisfy the equation

$$A |\alpha, f\rangle = \alpha |\alpha, f\rangle, \quad \alpha \in \mathbf{C}. \quad (61)$$

Looking for the decomposition of $|\alpha, f\rangle$ in the Fock space

$$|\alpha, f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad (62)$$

where $|n\rangle$, eigenfunction of \hat{n} , is a normalized Fock state, we obtain

$$c_n = N_{f,\alpha} \frac{\alpha^n}{\sqrt{n!} [f(n)]!}, \quad (63)$$

in which

$$[f(n)]! = f(0)f(1) \cdots f(n) \quad (64)$$

and

$$N_{f,\alpha} = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n! |[f(n)]!|^2} \right)^{-1/2}. \quad (65)$$

The scalar product is easily written

$$\langle \alpha, f | \beta, f \rangle = N_{f,\alpha} N_{f,\beta} \sum_{n=0}^{\infty} \frac{1}{n! |[f(n)]!|^2} (\alpha^* \beta)^n. \quad (66)$$

It should be remarked furthermore that, given $C(n) = C_n$ any real function on \mathbf{Z}^+ , the state $|\alpha, C\rangle$ defined by

$$|\alpha, C\rangle = \sum_{n=0}^{\infty} C_n \alpha^n |n\rangle \quad (67)$$

is an eigenfunction of some A . In fact, the corresponding function f is found to be

$$f(n) = \frac{1}{\sqrt{n}} \frac{C_{n-1}}{C_n}. \quad (68)$$

In case $f(n) = 1$, we have standard coherent states.

The known q-coherent states [3] turn out to be a particular case of f-coherent states. The normalization factor of such states is

$$N_{q,\alpha} = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-1/2}, \quad (69)$$

in which

$$[n]! = \frac{\sinh \lambda n}{\sinh \lambda} \frac{\sinh \lambda(n-1)}{\sinh \lambda} \dots 1. \quad (70)$$

For the scalar product, we have

$$\langle \alpha | \beta \rangle_{\lambda} = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!} \right)^{-1/2} \left(\sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[n]!} \right)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{[n]!} (\alpha^* \beta)^n. \quad (71)$$

7 Deformation of Planck formula

We will discuss what physical consequences may be found if the considered f-nonlinearity influences the vibrations of the real field mode oscillators like, for example, electromagnetic field oscillators or the oscillations of the nuclei in polyatomic molecules.

First it will be seen that this nonlinearity changes the specific heat behaviour. To show this, we have to find the partition function for a single f-oscillator corresponding to the Hamiltonian $H = (AA^{\dagger} + A^{\dagger}A)/2$, which is

$$Z(T) = \sum_{n=0}^{n=\infty} \exp(-\beta E_n), \quad (72)$$

where the variable β is the inverse temperature T^{-1} . Calculating the partition function for an ensemble of q-oscillators [1], we obtain that the specific heat decreases for $T \rightarrow \infty$ as

$$C \propto \frac{1}{\ln T}. \quad (73)$$

Thus, the behaviour of the specific heat of the q -oscillator is different from the behaviour of the usual oscillator in the high temperature limit. This property may serve as an experimental check of the existence of vibrational nonlinearity of the q -oscillator fields.

q -deformed Bose distribution can be obtained by the same method and one has for small q -nonlinearity parameter the following q -deformed Planck distribution formula [10]

$$\langle n \rangle = \frac{1}{e^{\hbar\omega/kT} - 1} - \lambda^2 \frac{\hbar\omega}{kT} \frac{e^{3\hbar\omega/kT} + 4e^{2\hbar\omega/kT} + e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT} - 1)^4}. \quad (74)$$

It means that q -nonlinearity deforms the formula for the mean photon numbers in black body radiation [1].

For small temperature, the behaviour of the deformed Planck distribution differs from the usual one [10]

$$\langle n \rangle - \bar{n}_0 = -\lambda^2 \frac{\hbar\omega}{kT} e^{-\hbar\omega/kT}. \quad (75)$$

As it was suggested in [9], the q -nonlinearity of the field vibrations produces blue shift effect which is the effect of the frequency increase with the field intensity. For small nonlinearity parameter λ and for large number of photons n in a given mode, the relative shift of the light frequency is

$$\frac{\delta\omega}{\omega} = \frac{\lambda^2 n^2}{2}.$$

This consequence of the possible existence of a q -nonlinearity may be relevant for models of the early stage of the Universe.

Another possible phenomenon related to the q -nonlinearity was considered in [11] where it was shown that if one deforms the electrostatics equation using the method of deformed creation and annihilation operators a point charge acquires a formfactor due to q -nonlinearity.

8 Conclusions

Starting with the example of the harmonic oscillator we have exhibited a family of associated nonlinear system which are completely integrable, both in classical and quantum physics.

We have shown that q -nonlinearity, associated with the Heisenberg–Weyl quantum groups, is a subcase of a more general class of possible nonlinearity.

A class of states has been considered in the Fock space through the deformation process applied to the harmonic oscillator operators. Such states have been described as f -coherent states, or nonlinear coherent states, and q -coherent states being particular examples of them.

The studied nonlinearities, if they exist, for the electromagnetic field or for the gluons, may influence the particle decays, correlations in particle multiplicities.

The deformation of the standard oscillator was interpreted as describing the behaviour of a nonlinear oscillator for which the frequency of vibrations depends on the amplitude of the vibrations. There is another physical aspect of this problem related to the change of quadrature commutation relations which, in fact, incorporate the influence of the nonlinearity of vibrations. If the oscillations of the quadratures are oscillations of a field, the change of the commutation relations means the different statistics of the field. Thus, from the introduced interpretation of the deformed oscillators [1, 9], it follows that the statistical properties of a field may depend on the intensity of the field. The experimental attempts to find the physical consequences of the changed Fermi or Bose statistics of different types have been done in [18–20].

As we have shown, the dynamical equation for quantum observable does not fix commutation relations and Hamiltonian in complete analogy with classical dynamical equations which may be described using different pairs: Hamiltonian–Poisson brackets. It means that the same quantum

evolution equation for a physical observable may be compatible with different pairs: quantum Hamiltonian–commutation relations for the observable. Physical consequence of the found result for quantum observable evolution is the following. If one wants to find out commutation relation of the physical observables using only known quantum dynamics for these observables, the quantum dynamics contains incomplete information to solve the problem and it yields only some constraints for the commutation relations.

The complete information to fix the commutation relations is contained in additional physical conditions of measurement procedure for measuring the observable. Thus, measuring the quadrature components gives experimentally established uncertainty relation. The result found in the work shows that the same quantum dynamics may exist both for observables which additionally satisfy the uncertainty relation for the oscillator quadratures and for the observables which do not satisfy the uncertainty relation since the commutation relations for these observables are different from standard position and momentum commutation relations.

Thus, with the quantum mechanics formalism are compatible different kinds of the same vibrational motion. For standard one, the quadratures satisfy in addition the uncertainty inequality. For others, they may not satisfy this inequality. Only physical nature of the vibrating quadratures which is determined by measurements distinguishes the two equally evolving, in the form of vibrations, quantum observables. This observation answers the question posed by Wigner: up to what extent the quantum equations of motion determine the quantum mechanics commutation relations? We have shown that there is an essential ambiguity in choosing the commutation relations which may be reduced only taking into account the measurement procedure for the observables.

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